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An action variable of the sine–Gordon model

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Abstract

It was conjectured that the classical bosonic string in AdS times a sphere has a special action variable which corresponds to the length of the operator on the field theory side. We discuss the analogous action variable in the sine–Gordon model. We explain the relation between this action variable and the Bäcklund transformations and show that the corresponding hidden symmetry acts on breathers by shifting their phase. It can be considered a nonlinear analogue of splitting the solution of the free field equations into positive- and negative-frequency parts.

Keywords: AdS/CFT correspondence; Classical integrable models; Extremal surfaces

1. Introduction

Studies of classical strings in $AdS_5 \times S^5$ formed an important part of the recent work on the AdS/CFT correspondence. It was observed that the energies of the fast moving classical strings reproduce the anomalous dimension of the field theory operators with the large R-charge, at least in the first and probably the second order of the perturbation theory [1–8].

The classical superstring in $AdS_5 \times S^5$ is an integrable system. An important tool in the study of this system is the super-Yangian symmetry discussed in [9,10]. The nonabelian dressing symmetries were also found in the classical Yang–Mills theory, see the recent discussion in [11] and the references therein. It was conjectured in [12–14] that the super-Yangian symmetry is also a symmetry of the Yang–Mills perturbation theory. It was shown that the one-loop anomalous

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dimension is proportional to the first Casimir operator in the Yangian representation. It is natural to conjecture that the higher loop contributions to the anomalous dimension correspond to the higher Casimirs in the Yangian representation, in the following sense. We conjecture that there are infinitely many operators $C_1, C_2, C_3, \ldots$ acting on the spin chain Hilbert space, commuting:

$$[C_i, C_j] = 0$$

and depending on the 'tHooft coupling constant λ as a power series:

$$C_j = C_{j,0} + \lambda C_{j,1} + \lambda^2 C_{j,2} + \cdots$$

and such that the anomalous dimension Δ has an expansion:

$$\Delta = \lambda C_1 + \lambda^2 C_2 + \lambda^3 C_3 + \cdots. \tag{1}$$

It should be true that $C_{j,0}$ involves j nearest neighbors in the chain. Notice that C_j explicitly depends on λ. Therefore the expansion of the anomalous dimension is:

$$\Delta = \lambda \Delta_1 + \lambda^2 \Delta_2 + \lambda^3 \Delta_3 + \cdots$$

where

$$\Delta_j = \sum_{k=0}^{j-1} C_{j-k,k}$$

Notice that $[\Delta_j, \Delta_k] \neq 0$ for $j \neq k$.

We have argued in [15] that a relation analogous to (1) holds for the classical string in $AdS_5 \times S^5$, the conserved charge $\lambda^j C_j$ corresponding to the j-th improved Pohlmeyer charge. We suggested identifying the anomalous dimension with the deck transformation acting on the phase space of the classical string in AdS times a sphere. The deck transformation can be defined as the action of the center of the conformal group. It is a *geometric* symmetry of the classical string; it comes from a geometric symmetry of the AdS space. But it can be expressed in terms of the *hidden* symmetries. String theory in AdS times a sphere has an infinite family of local conserved charges, the Pohlmeyer charges [16]. These Pohlmeyer charges can be thought of as the classical limit of the Yangian Casimirs. We have argued in [15] that the deck transformation is in fact generated by an infinite linear combination of the Pohlmeyer charges; the coefficients of this linear combination were fixed in [17]. We used in our arguments the existence of a special action variable in the theory of the classical string on S^n which was discussed in [18] following [19–21]. The special property of this particular action variable is that in each order of the null-surface perturbation theory [22,23] it is given by a local expression.[1] In each order of the perturbation theory we can approximate this action variable by a finite sum of the Pohlmeyer charges. This action variable corresponds to the *length* of the spin chain on the field theory side [27].

[1] The results of [24–26] imply that this property of the action variable does not hold for the superstring. Indeed, the construction of the action variable in Section 4 of [18] used the fact that the classical bosonic sigma-model splits into the AdS_5 part and the S^5 part. But the fermions "glue together" the AdS and the sphere. It was argued in [24–26] that this action variable still exists in the supersymmetric case but is not local. What survives the supersymmetric extension is the statement that the deck transformation in each order of the null-surface perturbation theory is generated by a finite sum of the local conserved charges (the classical analogues of the super-Yangian Casimirs). Indeed, the definition of the deck transformation does not require the splitting of the sigma-model into two parts and the locality of the deck transformation in the perturbation theory is manifest; it is essentially a consequence of the worldsheet causality. I want to thank N. Beisert for a discussion of this subject.

This "length" was studied for the finite gap solutions in [28,25]. Here we will study it for the rational solutions. We will consider the simplest case of the classical string on $\mathbf{R}\times S^2$. This system is related [16] to the sine–Gordon model and we will actually discuss mostly the sine–Gordon model. The existence of the special action variable can be understood locally on the worldsheet, at least in the null-surface perturbation theory. Therefore to study this action variable we do not have to impose the periodicity conditions on the spatial direction of the string worldsheet; we can formally consider infinitely long strings. This allows us to use the rational solutions of the sine–Gordon equation which are probably "simpler" than the finite gap solutions studied in [29,30,28,31,25] (at least if we consider the elementary functions "simpler" than the theta functions).

In Section 2 we discuss the relation between the classical string propagating on $\mathbf{R} \times S^2$ and the sine–Gordon model. In Section 3 we discuss the tau-function and bilinear identities. In Section 4.1 we discuss Bäcklund transformations and define the "hidden" symmetry $U(1)_L$. In Section 4.2 we consider the plane wave limit. In Section 4.3 we explain how $U(1)_L$ acts on breathers. In Section 4.4 we discuss the "improved" currents and show that $U(1)_L$ has a local expansion in the null-surface perturbation theory. In Section 5 we summarize our construction of the action variable and outline the analogous construction for the $O(N)$ sigma-model.

2. Sine–Gordon and string on $\mathbf{R} \times S^2$

2.1. Sine–Gordon equation from the classical string

The sine–Gordon model is one of the simplest exactly solvable models of interacting relativistic fields, and the bosonic string propagating on $\mathbf{R} \times S^2$ is one of the simplest nonlinear string worldsheet theories. On the level of classical equations of motion these two models are equivalent.

Consider the classical string propagating on $\mathbf{R} \times S^2$. Let t denote the time coordinate parametrizing $\mathbf{R}$. We will choose the conformal coordinates (τ, σ) on the worldsheet so that the induced metric is proportional to $\mathrm{d}\tau^2 - \mathrm{d}\sigma^2$. We will also fix the residual freedom in the choice of the conformal coordinates by putting $\tau = t$. We can parametrize the sphere by unit vectors $\vec{n}$; the embedding of the classical string in S^2 is parametrized by $\vec{n}(\tau, \sigma)$. The worldsheet equations of motion are:

$$(\partial_\tau^2 - \partial_\sigma^2)\vec{n} = -[(\partial_\tau \vec{n})^2 - (\partial_\sigma \vec{n})^2]\vec{n}. \tag{2}$$

These equations of motion follow from the constraints:

$$\left(\frac{\partial \vec{n}}{\partial \tau}\right)^2 + \left(\frac{\partial \vec{n}}{\partial \sigma}\right)^2 = 1 \tag{3}$$

$$\left(\frac{\partial \vec{n}}{\partial \tau}, \frac{\partial \vec{n}}{\partial \sigma}\right) = 0. \tag{4}$$

The map to the sine–Gordon model is given by [16]:

$$\cos 2\phi = \left(\frac{\partial \vec{n}}{\partial \tau}\right)^2 - \left(\frac{\partial \vec{n}}{\partial \sigma}\right)^2. \tag{5}$$

In other words

$$|\partial_\tau \vec{n}| = |\cos\phi|, \quad |\partial_\sigma \vec{n}| = |\sin\phi|. \tag{6}$$

The Virasoro constraints (3) are equivalent to the sine–Gordon equation:

$$\left[\partial_\tau^2 - \partial_\sigma^2\right]\phi = -\frac{1}{2}\sin 2\phi. \tag{7}$$

What can we say about the inverse map, from ϕ to $\vec{n}$? Let us consider the limit when the string moves very fast.

2.2. *Null-surface limit, plane wave limit, free field limit*

When the string moves very fast $|\partial_\tau \vec{n}| >> |\partial_\sigma \vec{n}|$. As in [22,23] we replace σ with $s = \epsilon\sigma$ where ϵ is a small parameter. Because of (6) we should also replace ϕ with $\epsilon\psi$; the new field $\psi(\tau, s)$ will be finite in the null-surface limit:

$$\sigma = \epsilon^{-1}s, \quad \phi(\tau,\sigma) = \epsilon\psi(\tau, s). \tag{8}$$

The string embedding $\vec{n}$ satisfies:

$$|\partial_s \vec{n}| = \psi - \frac{\epsilon^2}{6}\psi^3 + \cdots \tag{9}$$

$$|\partial_\tau \vec{n}| = 1 - \frac{\epsilon^2}{2}\psi^2 + \cdots. \tag{10}$$

The rescaled sine–Gordon field ψ satisfies:

$$[\partial_\tau^2 - \epsilon^2\partial_s^2]\psi = -\psi + \frac{\epsilon^2}{6}\psi^3 + \cdots. \tag{11}$$

In the strict null-surface limit $\epsilon = 0$ and

$$\psi(\tau, s) = a(s)\cos(\tau + \alpha(s)). \tag{12}$$

In this limit the string worldsheet is a collection of null-geodesics. The S^2-part is therefore a collection of equators of S^2. For each point (τ_0, s_0) on the worldsheet the intersection of S^2 with the 2-plane generated by $\vec{n}(\tau_0, s_0)$ and $\partial_\tau\vec{n}(\tau_0, s_0)$ is the corresponding equator; this equator can be parametrized by the vector $\vec{V} = [\vec{n} \times \partial_\tau \vec{n}]$. We have

$$\vec{n}(\tau, s_0) = \cos(\tau - \tau_0)\vec{n}(\tau_0, s_0) + \sin(\tau - \tau_0)\partial_{\tau_0}\vec{n}(\tau_0, s_0).$$

Eqs. (9) and (12) show that the one-parameter family of equators forming the null-surface is given by the equation

$$\partial_s \vec{V}(s) = [(-a(s)\vec{n}\cos(\tau_0 + \alpha(s)) + a(s)\partial_\tau\vec{n}\sin(\tau_0 + \alpha(s))) \times \vec{V}(s)]$$

where $\alpha(s)$ and $a(s)$ are determined from $\psi(\tau, s)$ by (12). Therefore in the limit $\epsilon \to 0$ the null-surface is determined by $\lim_{\epsilon\to 0}\epsilon^{-1}\phi$. It should be possible in principle to extend this analysis to higher orders and find the extremal surface corresponding to the solution ϕ of the sine–Gordon equation. The extremal surface is determined by ϕ up to the rotations of S^2.

Another important limit is the *plane wave limit*. To get to the plane wave limit we first go to the null-surface limit (8) and then take an additional rescaling $\psi = \epsilon_1\chi$. In the strict limit $\epsilon_1 = 0$ the equations for χ become linear:

$$[\partial_\tau^2 - \epsilon^2\partial_s^2]\chi = -\chi. \tag{13}$$

The plane wave limit is therefore the free field limit.

2.3. Poisson structure

The canonical Poisson structure of the classical string on $\mathbf{R} \times S^2$ does not agree with the canonical Poisson structure of the sine–Gordon model. But it corresponds to another Poisson structure of the sine–Gordon model, which is compatible with the canonical one [32]. Therefore, the Hamiltonian flows generated by the local conserved charges of the sine–Gordon model should differ from the flows of the Pohlmeyer charges of the classical string only by the relabelling of the charges. This means that the action variable of the sine–Gordon model which we discuss in this paper corresponds to the action variable of the classical string, which is also generated by an infinite linear combination of the local Pohlmeyer charges.

3. Rational solutions

3.1. Tau-functions and the dependence on higher times

In this section we will discuss the dependence of the sine–Gordon solutions on the "higher times" following mostly [33–35]. We will first introduce the tau-functions and then explain how they are related to the solutions of the sine–Gordon equations.

The tau-functions for the rational solutions are

$$\tau_\pm = \det(1 \pm \mathcal{V}) \tag{14}$$

$$\begin{aligned} \mathcal{V}_{jk} &= 2ib_jb_k\frac{\sqrt{\lambda_j\lambda_k}}{\lambda_j+\lambda_k} \\ &\quad\times\exp\left[\sum_p t_{2p+1}\left(\lambda_j^{2p+1}+\lambda_k^{2p+1}\right)-\sum_p \tilde{t}_{2p+1}\left(\lambda_j^{-2p-1}+\lambda_k^{-2p-1}\right)\right]. \end{aligned} \tag{15}$$

Here b_j and λ_j, $j = 1, \ldots, N$, are parameters characterizing the solution, and $t_{2p+1}, \tilde{t}_{2p+1}$, $p = 0, 1, 2, \ldots$, are the so-called times. We identify $t_1 = \frac{1}{4}(\tau + \sigma)$ and $\tilde{t}_1 = \frac{1}{4}(\tau - \sigma)$. The "higher" times $t_3, \tilde{t}_3, t_5, \tilde{t}_5, \ldots$ correspond to the higher conserved charges. Changing the higher times corresponds to the motion on the "Liouville torus" in the phase space. Rational solutions correspond to finite N; the tau-functions of the rational solutions are the determinants of the $N \times N$ matrices $\delta_{ij} \pm \mathcal{V}_{ij}$.

Let us consider the left and right Bäcklund transformations:

$$B_\mu.\tau_\pm(\{t_{2p+1}\},\{\tilde{t}_{2q+1}\}) = \tau_\pm\left(\left\{t_{2p+1}-\frac{\mu^{-2p-1}}{2p+1}\right\},\left\{\tilde{t}_{2q+1}\right\}\right) \tag{16}$$

$$\tilde{B}_{\tilde{\mu}}.\tau_\pm(\{t_{2p+1}\},\{\tilde{t}_{2q+1}\}) = \tau_\pm\left(\left\{t_{2p+1}\right\},\left\{\tilde{t}_{2q+1}+\frac{\tilde{\mu}^{2q+1}}{2q+1}\right\}\right) \tag{17}$$

where μ and $\tilde{\mu}$ are constant parameters. The tau-functions satisfy the following bilinear identities:

$$\begin{aligned} &B_\mu B_\nu\tau_+\tau_- + \tau_+B_\mu B_\nu\tau_- = B_\mu\tau_+B_\nu\tau_- + B_\nu\tau_+B_\mu\tau_- \\ &\frac{\nu-\mu}{\nu+\mu}(B_\mu B_\nu\tau_+\tau_- - \tau_+B_\mu B_\nu\tau_-) = B_\mu\tau_+B_\nu\tau_- - B_\nu\tau_+B_\mu\tau_- \end{aligned} \tag{18}$$

$$\begin{aligned}
&\tilde{B}_{\tilde{\mu}}\tilde{B}_{\tilde{\nu}}\tau_+\tau_- + \tau_+\tilde{B}_{\tilde{\mu}}\tilde{B}_{\tilde{\nu}}\tau_- = \tilde{B}_{\tilde{\mu}}\tau_+\tilde{B}_{\tilde{\nu}}\tau_- + \tilde{B}_{\tilde{\nu}}\tau_+\tilde{B}_{\tilde{\mu}}\tau_-\\
&\frac{\tilde{\nu}-\tilde{\mu}}{\tilde{\nu}+\tilde{\mu}}(\tilde{B}_{\tilde{\mu}}\tilde{B}_{\tilde{\nu}}\tau_+\tau_- - \tau_+\tilde{B}_{\tilde{\mu}}\tilde{B}_{\tilde{\nu}}\tau_-) = -\tilde{B}_{\tilde{\mu}}\tau_+\tilde{B}_{\tilde{\nu}}\tau_- + \tilde{B}_{\tilde{\nu}}\tau_+\tilde{B}_{\tilde{\mu}}\tau_-
\end{aligned} \tag{19}$$

$$\begin{aligned}
&B_\mu\tilde{B}_{\tilde{\nu}}\tau_+\tau_+ + B_\mu\tilde{B}_{\tilde{\nu}}\tau_-\tau_- = B_\mu\tau_+\tilde{B}_{\tilde{\nu}}\tau_+ + B_\mu\tau_-\tilde{B}_{\tilde{\nu}}\tau_-\\
&\frac{\tilde{\nu}-\mu}{\tilde{\nu}+\mu}(B_\mu\tilde{B}_{\tilde{\nu}}\tau_-\tau_- - \tau_+B_\mu\tilde{B}_{\tilde{\nu}}\tau_+) = B_\mu\tau_+\tilde{B}_{\tilde{\nu}}\tau_+ - \tilde{B}_{\tilde{\nu}}\tau_-B_\mu\tau_-.
\end{aligned} \tag{20}$$

These bilinear identities can be derived from the free fermion representation of the tau-function as explained for example is [35]. We introduce free fermions $\psi(\mu) = \sum_{m\in\mathbf{Z}}\psi_m\mu^{m-1/2}$ and $\tilde{\psi}(\mu) = \sum_{m\in\mathbf{Z}}\tilde{\psi}_m\mu^{-m+1/2}$, $\{\psi_m, \tilde{\psi}_n\} = \delta_{mn}$. The "vacuum vectors" are labelled by $k \in \mathbf{Z}$ so that $\langle k|\psi(\lambda_1)\tilde{\psi}(\lambda_2)|k\rangle = (\frac{\lambda_1}{\lambda_2})^k\frac{\sqrt{\lambda_1\lambda_2}}{\lambda_1-\lambda_2}$ for $|\lambda_1| > |\lambda_2|$. Let us put $k_+ = 0$ and $k_- = 1$. We have

$$\begin{aligned}
\tau_\pm = {}& \mathrm{e}^{-\sum(2p+1)t_{2p+1}\tilde{t}_{2p+1}}\\
&\times\left\langle k_\pm|\mathrm{e}^{\sum t_{2p+1}\psi_n\tilde{\psi}_{n+2p+1}}\prod_{j=1}^N\left[1+2b_j^2\psi(\lambda_j)\tilde{\psi}(-\lambda_j)\right]\mathrm{e}^{\sum\tilde{t}_{2p+1}\psi_n\tilde{\psi}_{n-2p-1}}|k_\pm\right\rangle.
\end{aligned} \tag{21}$$

Eq. (18) is Eq. (2.42) of [35] if we take into account that

$$\tau_- = \lim_{\mu\to 0} B_\mu\tau_+. \tag{22}$$

Let us study some differential equations following from the bilinear identities. From (20) we have at the first order in $\tilde{\nu}/\mu$:

$$\tau_-\partial_{t_1}\partial_{\tilde{t}_1}\tau_- - \partial_{t_1}\tau_-\partial_{\tilde{t}_1}\tau_- = -\tau_-^2 + \tau_+^2 \tag{23}$$

$$\tau_+\partial_{t_1}\partial_{\tilde{t}_1}\tau_+ - \partial_{t_1}\tau_+\partial_{\tilde{t}_1}\tau_+ = -\tau_+^2 + \tau_-^2. \tag{24}$$

Therefore the equations of motion for the sine–Gordon model

$$\frac{\partial^2}{\partial t_1\partial\tilde{t}_1}\phi = -2\sin 2\phi \tag{25}$$

follow if we set

$$\phi = i\log\frac{\tau_+}{\tau_-}. \tag{26}$$

Expanding (18) in powers of $\frac{1}{\nu}$ we have

$$\partial_{t_1}\tau_-B_\mu\tau_+ - \tau_-\partial_{t_1}B_\mu\tau_+ = \mu(\tau_-B_\mu\tau_+ - \tau_+B_\mu\tau_-) \tag{27}$$

and the same equation with τ_+ and τ_- exchanged. This can be rewritten as the first order differential equation relating $B_\mu\phi$ to ϕ:

$$\frac{\partial}{\partial t_1}(B_\mu\phi + \phi) = -2\mu\sin(B_\mu\phi - \phi). \tag{28}$$

Expanding (20) in powers of $\tilde{\nu}$ we get:

$$\tau_+\partial_{\tilde{t}_1}B_\mu\tau_+ - \partial_{\tilde{t}_1}\tau_+B_\mu\tau_+ = \frac{1}{\mu}(\tau_+B_\mu\tau_+ - \tau_-B_\mu\tau_-) \tag{29}$$

and the same equation with τ_+ exchanged with τ_-. This gives us the second equation relating $B_\mu\phi$ to ϕ:

$$\frac{\partial}{\partial \tilde{t}_1}(B_\mu\phi - \phi) = \frac{2}{\mu}\sin(B_\mu\phi + \phi). \tag{30}$$

Expanding (19) in powers of $\tilde{\nu}$ we get

$$\tau_-\partial_{\tilde{t}_1}\tilde{B}_{\tilde{\mu}}\tau_+ - \tilde{B}_{\tilde{\mu}}\tau_+\partial_{\tilde{t}_1}\tau_- = \frac{1}{\tilde{\mu}}(\tau_-\tilde{B}_{\tilde{\mu}}\tau_+ - \tau_+\tilde{B}_{\tilde{\mu}}\tau_-) \tag{31}$$

and the same equation with τ_+ and τ_- exchanged. This gives us the equation relating $\tilde{B}_{\tilde{\mu}}\phi$ to ϕ:

$$\frac{\partial}{\partial \tilde{t}_1}(\tilde{B}_{\tilde{\mu}}\phi + \phi) = \frac{2}{\tilde{\mu}}\sin(\tilde{B}_{\tilde{\mu}}\phi - \phi). \tag{32}$$

The second equation follows from (20):

$$\frac{\partial}{\partial t_1}(\tilde{B}_{\tilde{\mu}}\phi - \phi) = -2\tilde{\mu}\sin(\tilde{B}_{\tilde{\mu}}\phi + \phi). \tag{33}$$

Eqs. (28), (30), (32) and (33) are usually taken as the definition of the left and right Bäcklund transformations. These equations do not determine $B_\mu\phi$ and $\tilde{B}_{\tilde{\mu}}\phi$ unambiguously from ϕ because there are integration constants. Eqs. (16) and (17) provide a particular solution.

The Bäcklund transformations for the sine–Gordon field correspond to the Bäcklund transformations for the classical string. If $\vec{n}(\tau,\sigma)$ is a string worldsheet and ϕ is the corresponding solution of the sine–Gordon model defined by Eq. (5) then

$$B_\mu\vec{n} = \frac{1-\mu^{-2}}{1+\mu^{-2}}\vec{n} - \frac{\mu^{-1}}{1+\mu^{-2}}\left(\frac{\sin(\phi - B_\mu\phi)}{\sin(2\phi)}\partial_{\tilde{t}_1}\vec{n} + \frac{\sin(\phi + B_\mu\phi)}{\sin(2\phi)}\partial_{t_1}\vec{n}\right) \tag{34}$$

satisfies

$$\begin{aligned}\partial_{\tilde{t}_1}(B_\mu\vec{n} - \vec{n}) &= -\frac{1}{2}(1+\mu^{-2})(B_\mu\vec{n}, \partial_{\tilde{t}_1}\vec{n})(B_\mu\vec{n} + \vec{n})\\ \partial_{t_1}(B_\mu\vec{n} + \vec{n}) &= \frac{1}{2}(1+\mu^{2})(B_\mu\vec{n}, \partial_{t_1}\vec{n})(B_\mu\vec{n} - \vec{n})\end{aligned} \tag{35}$$

and

$$\tilde{B}_{\tilde{\mu}}\vec{n} = \frac{1-\tilde{\mu}^{2}}{1+\tilde{\mu}^{2}}\vec{n} + \frac{\tilde{\mu}}{1+\tilde{\mu}^{2}}\left(\frac{\sin(\phi - \tilde{B}_{\tilde{\mu}}\phi)}{\sin(2\phi)}\partial_{t_1}\vec{n} + \frac{\sin(\phi + \tilde{B}_{\tilde{\mu}}\phi)}{\sin(2\phi)}\partial_{\tilde{t}_1}\vec{n}\right) \tag{36}$$

satisfies

$$\begin{aligned}\partial_{t_1}(\tilde{B}_{\tilde{\mu}}\vec{n} - \vec{n}) &= -\frac{1}{2}(1+\tilde{\mu}^{2})(\tilde{B}_{\tilde{\mu}}\vec{n}, \partial_{t_1}\vec{n})(\tilde{B}_{\tilde{\mu}}\vec{n} + \vec{n})\\ \partial_{\tilde{t}_1}(\tilde{B}_{\tilde{\mu}}\vec{n} + \vec{n}) &= \frac{1}{2}(1+\tilde{\mu}^{-2})(\tilde{B}_{\tilde{\mu}}\vec{n}, \partial_{\tilde{t}_1}\vec{n})(\tilde{B}_{\tilde{\mu}}\vec{n} - \vec{n}).\end{aligned} \tag{37}$$

The relation between $B_\mu\vec{n}$ and $B_\mu\phi$, and between $\tilde{B}_{\tilde{\mu}}\vec{n}$ and $\tilde{B}_{\tilde{\mu}}\phi$, is given by Eq. (5). The relation between Bäcklund transformations in the $O(3)$ model and in the sine–Gordon model has been previously discussed in [36].

3.2. The reality conditions and a restriction on the class of solutions

To get the real solutions of the sine–Gordon theory we need τ_+ to be the complex conjugate of τ_-. This can be achieved if the parameters λ_i come in pairs λ_k and λ_{N-k} such that $\lambda_k = \overline{\lambda}_{N-k}$ and $b_k = \overline{b}_{N-k}$. We want to restrict ourselves to considering only the solutions for which all λ_j have a nonzero imaginary part:

$$\mathrm{Im}\,\lambda_j \neq 0. \tag{38}$$

The purely real λ_j would lead to kinks; we consider the solutions with kinks too far from being the fast moving strings.

General solutions of the sine–Gordon equations on a real line were discussed in [34] using the inverse scattering method. There is a difference in notation: our λ_j differ from λ_j of [34] by a factor of i. The scattering data of the general solution includes a discrete set of real (in our notation) $\lambda_j = \kappa_j$, $\kappa_j \in \mathbf{R}$. Besides that, there is a discrete set of complex conjugate pairs $(\lambda_j, \overline{\lambda}_j)$ with $\mathrm{Im}\ \lambda_j \neq 0$ and also a continuous data parametrized by a function $b(x)$ with $\overline{b}(x) = b(-x)$. General solutions can be approximated by the rational solutions, which have $b(x) = 0$. Therefore rational solutions depend only on the discrete set of parameters κ_j and $(\lambda_k, \overline{\lambda}_k)$. It is useful to look at the asymptotic form of these rational solutions in the infinite future, when $t = t_1 + \tilde{t}_1 = \infty$. At $t = \infty$ the rational solutions split into well-separated breathers (corresponding to $(\lambda_k, \overline{\lambda}_k)$) and kinks (corresponding to κ_j). The energy of a breather can be made very small by putting λ_k sufficiently close to the imaginary axis (see Section 4.3). This means that one can continuously create a new breather from the vacuum. In other words, creation of the new pair $(\lambda_k\overline{\lambda}_k)$ is a continuous operation; it changes a solution in the continuous way. But the creation of a kink is not a continuous operation. The creation of an odd number of kinks would necessarily change the topological charge of the solution. But even to create a pair of kink and anti-kink would require a finite energy. This is our justification for considering separately a sector of solutions which do not have real λ_j. We will discuss the action variable in this sector.

4. Bäcklund transformations and the "hidden" symmetry $U(1)_L$

4.1. Construction of $U(1)_L$

Eq. (16) shows that the Bäcklund transformations[2] can be understood as a μ-dependent shift of times. We have two one-parameter families of shifts B_μ and $\tilde{B}_{\tilde{\mu}}$. We have $B_{\mu=\infty} = \mathbf{1}$ and $\tilde{B}_{\tilde{\mu}=0} = \mathbf{1}$. It is not true that B_μ or $\tilde{B}_{\tilde{\mu}}$ is a one-parameter group of transformations, because it is not true that $B_{\mu_1}B_{\mu_2}$ is equal to B_{μ_3} with some μ_3. Both B_μ and $\tilde{B}_{\tilde{\mu}}$ preserve the symplectic structure. Therefore we can discuss the Hamiltonian vector fields ξ_μ and $\tilde{\xi}_{\tilde{\mu}}$ such that:

$$\mathrm{e}^{\xi_\mu} = B_\mu, \quad \mathrm{e}^{\tilde{\xi}_{\tilde{\mu}}} = \tilde{B}_{\tilde{\mu}}. \tag{39}$$

One could imagine an ambiguity in the definition of ξ_μ and $\tilde{\xi}_{\tilde{\mu}}$, but we have the continuous families connecting B_μ to $\mathbf{1} = B_\infty$ and $\tilde{B}_{\tilde{\mu}}$ to $\mathbf{1} = \tilde{B}_0$. The existence of these continuous families allows us to define ξ_μ and $\tilde{\xi}_{\tilde{\mu}}$ unambiguously, see Fig. 1. The formula is:

[2] More precisely, a particular solution of the Bäcklund equations, defined as a series in $1/\mu$ or $\tilde{\mu}$.

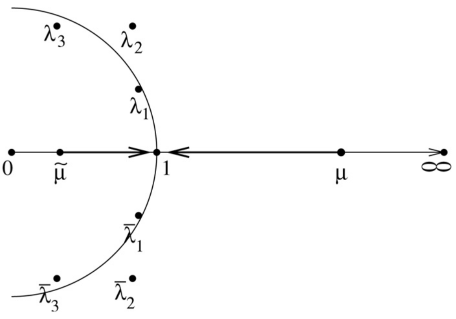

Fig. 1. The Bäcklund transformation B_μ can be expanded in $1/\mu$ near $\mu = \infty$ and $\tilde{B}_{\tilde{\mu}}$ can be expanded in $\tilde{\mu}$ near $\tilde{\mu} = 0$. We analytically continue B and $\tilde{B}$ to $\mu = \tilde{\mu} = 1$.

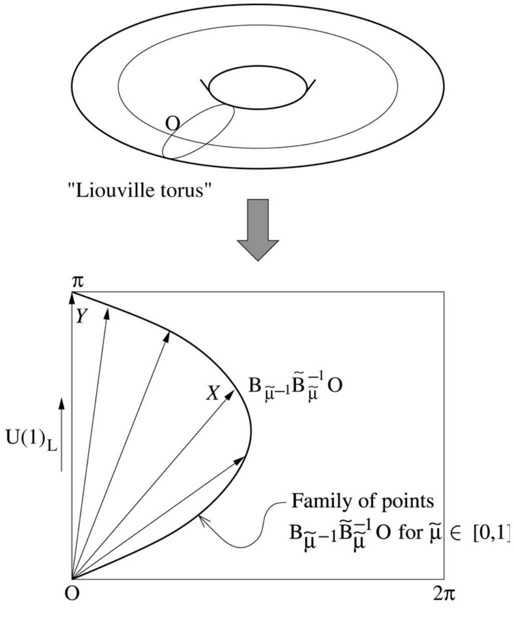

Fig. 2. The relation between the hidden symmetry $U(1)_L$ and the Bäcklund transformations. We pick a point O on a "Liouville torus" (for a finite-gap solution, this is an actual finite-dimensional torus). The solid curve OXY represents the one-parameter family of points $B_{\tilde{\mu}^{-1}}\tilde{B}_{\tilde{\mu}}^{-1}O$ parametrized by $\tilde{\mu} \in [0, 1]$. When $\tilde{\mu} = 1$ the interval OY (where $Y = B_1\tilde{B}_1^{-1}O$) is the b-cycle of the torus. The symmetry $U(1)_L$ acts by shifts along this cycle. The arrow OX represents the one-parameter family of points $\exp[t\log(B_{\tilde{\mu}^{-1}}\tilde{B}_{\tilde{\mu}}^{-1})]O$ with $t \in [0, 1]$. When $\tilde{\mu} \to 1$ the one-parameter group $\exp[t\log(B_1\tilde{B}_1^{-1})]$ with $t \in [0, 2]$ is $U(1)_L$.

$$\xi_\mu = -\sum_{p=0}^{\infty} \frac{\mu^{-2p-1}}{2p+1}\frac{\partial}{\partial t_{2p+1}} \tag{40}$$

$$\tilde{\xi}_{\tilde{\mu}} = \sum_{p=0}^{\infty} \frac{\tilde{\mu}^{2p+1}}{2p+1}\frac{\partial}{\partial \tilde{t}_{2p+1}}. \tag{41}$$

These vector fields act on the rational solutions through the parameters b_j:

$$\xi_\mu.b_j = \frac{1}{2}\log\left[\frac{1-\lambda_j/\mu}{1+\lambda_j/\mu}\right]b_j \tag{42}$$

$$\tilde{\xi}_{\tilde{\mu}}.b_j = \frac{1}{2}\log\left[\frac{1-\tilde{\mu}/\lambda_j}{1+\tilde{\mu}/\lambda_j}\right]b_j. \tag{43}$$

Let us consider the limit:

$$\xi = \lim_{\epsilon\to 0+}(\xi_{\mathrm{e}^\epsilon} - \tilde{\xi}_{\mathrm{e}^{-\epsilon}}) = \lim_{\epsilon\to 0+}\sum_{p=0}^{\infty}\left[-\frac{\mathrm{e}^{-(2p+1)\epsilon}}{2p+1}\frac{\partial}{\partial t_{2p+1}} - \frac{\mathrm{e}^{-(2p+1)\epsilon}}{2p+1}\frac{\partial}{\partial \tilde{t}_{2p+1}}\right]. \tag{44}$$

We have:

$$\xi.b_j = -\frac{\pi i}{2}\,\mathrm{sign}(\mathrm{Im}(\lambda_j))b_j. \tag{45}$$

We see that the trajectories of the vector field ξ are periodic:

$$\mathrm{e}^{2\xi} = \mathbf{1}. \tag{46}$$

Therefore ξ is the Hamiltonian vector field of an *action variable*. We denote as $U(1)_L$ the corresponding hidden symmetry. See Fig. 2. Notice that e^ξ exchanges τ_+ and τ_- and therefore maps $\phi \mapsto -\phi$:

$$\mathrm{e}^\xi = [\phi \mapsto -\phi]. \tag{47}$$

The corresponding symmetry of the classical string is $\vec{n} \mapsto -\vec{n}$. We see that the discrete *geometric* $\mathbf{Z}_2$-symmetry (the "reflection" $\phi \mapsto -\phi$) is related to the continuous *hidden* symmetry $U(1)_L$ (generated by the higher Hamiltonians). This example is of the same nature as the relation between the anomalous dimension and the local charges discussed in [15].

In the language of free fermions, B_μ corresponds to the creation of the free fermion $\psi(\mu)$ from the left vacuum, and $\tilde{B}_{\tilde{\mu}}^{-1}$ to the creation of $\tilde{\psi}(\tilde{\mu})$ from the right vacuum. When $\mu \to \tilde{\mu}$, the leading term in the operator product expansion of $\psi(\mu)\tilde{\psi}(\tilde{\mu})$ is a c-number, and it cancels between τ_+ and τ_- in (26). The shift of the charge of the left and right Dirac vacua leads to the exchange $\tau_+ \leftrightarrow \tau_-$, and therefore Eq. (26) gives $\phi \mapsto -\phi$.

This construction essentially used the fact that $B_1\tilde{B}_1^{-1} = [\phi \mapsto -\phi]$. In fact for any real $\mu = \tilde{\mu}$ we have

$$B_\mu \tilde{B}_{\tilde{\mu}}^{-1}|_{\tilde{\mu}=\mu} = [\phi \mapsto -\phi]. \tag{48}$$

This can be understood directly from (28), (30), (32) and (33). First of all we have to explain the meaning of the left hand side of (48), because we defined B_μ only for large μ as a series in $1/\mu$ and $\tilde{B}_{\tilde{\mu}}$ for small $\tilde{\mu}$ as a series in $\tilde{\mu}$. Let us consider the null-surface limit (8) and construct B_μ and $\tilde{B}_{\tilde{\mu}}$ as a series in ϵ, where ϵ is the small parameter of the null-surface perturbation theory defined in (8). In this perturbation theory we have $|\partial_\sigma\phi| << |\partial_\tau\phi|$ and $|\phi| << 1$. The zeroth approximation to (28), (30) and (32), (33) is:

$$B_\mu\phi = \frac{1-\mu^{-2}}{1+\mu^{-2}}\phi - \frac{2\mu^{-1}}{1+\mu^{-2}}\partial_\tau\phi \tag{49}$$

$$\tilde{B}_{\tilde{\mu}}\phi = \frac{1-\tilde{\mu}^2}{1+\tilde{\mu}^2}\phi - \frac{2\tilde{\mu}}{1+\tilde{\mu}^2}\partial_\tau\phi. \tag{50}$$

We see that in the leading order of the null-surface perturbation theory B_μ and $\tilde{B}_{\tilde{\mu}}$ both depend on μ and $\tilde{\mu}$ as rational functions. The higher orders are also rational functions of μ and $\tilde{\mu}$. Therefore in the null-surface perturbation theory B_μ and $\tilde{B}_{\tilde{\mu}}$ both have an unambiguous analytic continuation to finite values of μ and $\tilde{\mu}$. Therefore we can take $\mu = \tilde{\mu}$ and (48) follows from (28), (30), (32), (33) and (49), (50). This means that the generator of $U(1)_L$ which we defined in (44) as $\xi_1 - \tilde{\xi}_1$ can also be defined as $\xi_\mu - \tilde{\xi_\mu}$ for any real μ. For the rational solutions with all λ_j having a nonzero imaginary part we have

$$(\xi_\mu - \tilde{\xi}_\mu) - (\xi_1 - \tilde{\xi}_1) = 0 \tag{51}$$

for any $\mu \in \mathbf{R}$. Therefore $U(1)_L$ commutes with the Lorentz boosts which transform $\phi(t_1, \tilde{t}_1)$ to $\phi(\mu t_1, \mu^{-1}\tilde{t}_1)$.

4.2. *Free field limit*

In the limit $\phi \to 0$ the equations of motion become

$$\partial_{t_1}\partial_{\tilde{t}_1}\phi = -4\phi. \tag{52}$$

And the left and right Bäcklund transformations become:

$$B_\mu.\phi = \frac{1 - \frac{1}{2\mu}\frac{\partial}{\partial t_1}}{1 + \frac{1}{2\mu}\frac{\partial}{\partial t_1}}\phi \tag{53}$$

$$\tilde{B}_{\tilde{\mu}}.\phi = \frac{1 + \frac{\tilde{\mu}}{2}\frac{\partial}{\partial \tilde{t}_1}}{1 - \frac{\tilde{\mu}}{2}\frac{\partial}{\partial \tilde{t}_1}}\phi. \tag{54}$$

This means that in the free field limit:

$$\frac{\partial}{\partial t_{2p+1}}\phi = \frac{1}{2^{2p}}\left(\frac{\partial}{\partial t_1}\right)^{2p+1}\phi \tag{55}$$

$$\frac{\partial}{\partial \tilde{t}_{2p+1}}\phi = \frac{1}{2^{2p}}\left(\frac{\partial}{\partial \tilde{t}_1}\right)^{2p+1}\phi. \tag{56}$$

The generator of $U(1)_L$ acts as follows:

$$\begin{aligned}\xi.\phi &= \lim_{\epsilon\to 0+}\log\left[\frac{(2-\mathrm{e}^{-\epsilon}\partial_{t_1})(2-\mathrm{e}^{-\epsilon}\partial_{\tilde{t}_1})}{(2+\mathrm{e}^{-\epsilon}\partial_{t_1})(2+\mathrm{e}^{-\epsilon}\partial_{\tilde{t}_1})}\right]\phi \\ &= \lim_{\epsilon\to 0+}\log\left[\frac{4\sinh\epsilon - \partial_{t_1} - \partial_{\tilde{t}_1}}{4\sinh\epsilon + \partial_{t_1} + \partial_{\tilde{t}_1}}\right]\phi.\end{aligned} \tag{57}$$

The free field ϕ has an oscillator expansion:

$$\phi = \int \frac{\mathrm{d}k}{\sqrt{2\omega_k}}\left(\alpha_k \mathrm{e}^{ik\sigma + i\omega_k\tau} + \overline{\alpha_k}\mathrm{e}^{-ik\sigma - i\omega_k\tau}\right) \tag{58}$$

where $\omega_k = \sqrt{4+k^2}$. Eq. (57) implies that $U(1)_L$ is the oscillator number:

$$\xi.\alpha_k = \pi i \alpha_k$$
$$\xi.\overline{\alpha_k} = -\pi i \overline{\alpha_k}. \tag{59}$$

This is in agreement with the results of [17] and shows that the $U(1)_L$ considered here is the same $U(1)_L$ as was considered in [18,17].

4.3. Action of $U(1)_L$ on a breather

Consider $\lambda_1 = \lambda = i e^{i\theta}|\lambda|$, $\lambda_2 = \overline{\lambda}$, $b_1 = b e^{i\varphi}$ and $b_2 = b e^{-i\varphi}$ and define $e^{\kappa} = b^2/\tan\theta$. We get

$$\tau_\pm = \frac{2b^2}{\tan\theta} e^{-2t\sin\theta|\lambda|+2\tilde{t}\sin\theta|\lambda|^{-1}} \tag{60}$$
$$\times\Big[\cosh(2t|\lambda|\sin\theta - 2\tilde{t}|\lambda|^{-2}\sin\theta - \kappa)$$
$$\pm i\tan\theta\cos(2t|\lambda|\cos\theta + 2\tilde{t}|\lambda|^{-1}\cos\theta + 2\varphi)\Big]. \tag{61}$$

Therefore

$$\tan\frac{\phi}{2} = \tan\theta\frac{\cos(2t|\lambda|\cos\theta + 2\tilde{t}|\lambda|^{-1}\cos\theta + 2\varphi)}{\cosh(2t|\lambda|\sin\theta - 2\tilde{t}|\lambda|^{-1}\sin\theta - \kappa)}. \tag{62}$$

Remember that $t_1 = \frac{1}{4}(\tau+\sigma)$ and $\tilde{t}_1 = \frac{1}{4}(\tau-\sigma)$. The limit $\theta \to 0$ corresponds to a circular null-string. Indeed, with $|\lambda| = 1$ Eqs. (6) and (62) imply in this limit that at $\tau = 0$ we have $\int_{-\infty}^{\infty} d\sigma|\partial_\sigma\vec{n}| = 2\pi$.

The generator of $U(1)_L$ acts on a breather by shifting the phase φ:

$$\xi = \frac{\pi}{2}\mathrm{sign}(\cos\theta)\frac{\partial}{\partial\varphi}. \tag{63}$$

The general solution without kinks can be approximated by collections of breathers. The $U(1)_L$ will shift the phases of all the breathers by the same amount.

We have seen in Section 4.2 that the generator of $U(1)_L$ can also be understood as the nonlinear analogue of the oscillator number. On the other hand, we can see from Eqs. (62) and (63) that in the null-surface limit

$$\xi \simeq \frac{\pi}{4}\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial\tilde{t}}\right) + \cdots \tag{64}$$

where dots denote the terms subleading in the null-surface limit. The leading term is the energy of the string, and the subleading terms are the higher conserved charges. The fact that the energy is the oscillator number plus corrections was observed already in the work of H.J. de Vega, A.L. Larsen and N. Sanchez [37].[3]

[3] I want to thank A. Tseytlin for bringing this work to my attention.

4.4. The null-surface limit and the "improved" currents of [19,20]

Eq. (62) shows that in the null-surface limit the parameters λ_j are localized in the vicinity of $\pm i$:

$$\lambda_j = \pm i + O(\epsilon). \tag{65}$$

The action of the higher Hamiltonians on the parameters b_j follows from (15):

$$\frac{\partial}{\partial t_{2p+1}} b_j = \lambda_j^{2p+1} b_j. \tag{66}$$

Consider the following linear combination of the higher Hamiltonian vector fields:

$$\sum_{p=0}^{l} \frac{l!}{p!(l-p)!} \frac{\partial}{\partial t_{2p+1}} b_j = \lambda_j (\lambda_j - i)^l (\lambda_j + i)^l b_j \simeq \epsilon^l b_j. \tag{67}$$

We see that the vector fields

$$\Xi_l = \sum_{p=0}^{l} \frac{l!}{p!(l-p)!} \frac{\partial}{\partial t_{2p+1}} \tag{68}$$

are generated by the "improved" currents; the vector field Ξ_l is of the order ϵ^l in the null-surface perturbation theory.

The improved currents used in [19,20] involve both left and right times. Let us introduce the improved Hamiltonian vector fields $\mathcal{G}_k$ which acts on the parameters b_j in the following way:

$$\mathcal{G}_k . b_j = \left(\lambda_j - \frac{1}{\lambda_j}\right) \left[\left(\lambda_j + \frac{1}{\lambda_j}\right)\right]^{2k} b_j. \tag{69}$$

These vector fields are local and improved, in the sense that $\mathcal{G}_k \simeq \epsilon^{2k}$ in the null-surface limit (65). For example $\mathcal{G}_0 = \frac{\partial}{\partial t_1} + \frac{\partial}{\partial \tilde{t}_1}$, $\mathcal{G}_1 = \frac{\partial}{\partial t_3} + \frac{\partial}{\partial t_1} + \frac{\partial}{\partial \tilde{t}_1} + \frac{\partial}{\partial \tilde{t}_3}$ and $\mathcal{G}_2 = \frac{\partial}{\partial t_5} + 3\frac{\partial}{\partial t_3} + 2\frac{\partial}{\partial t_1} + 2\frac{\partial}{\partial \tilde{t}_1} + 3\frac{\partial}{\partial \tilde{t}_3} + \frac{\partial}{\partial \tilde{t}_5}$.

The Hamiltonian vector fields $\frac{\partial}{\partial t_{2p+1}} + \frac{\partial}{\partial \tilde{t}_{2p+1}}$ can be expressed through the improved vector fields:

$$\frac{\partial}{\partial t_{2p+1}} + \frac{\partial}{\partial \tilde{t}_{2p+1}} = \sum_{n=0}^{p} 2^{-2n} U_{2p,2n} \mathcal{G}_n \tag{70}$$

where $U_{2p,2n}$ are the coefficients of the Chebyshev polynomials of the second kind:

$$U_{2p}(x) = \sum_{k=0}^{p} U_{2p,2k} x^{2k} = \frac{(x + i\sqrt{1-x^2})^{2p+1} - (x - i\sqrt{1-x^2})^{2p+1}}{2i\sqrt{1-x^2}}. \tag{71}$$

In the null-surface perturbation theory $\mathcal{G}_k \simeq \epsilon^{2k}$. On the other hand, for λ sufficiently close to i or $-i$ we have

$$\frac{\pi i}{2} \operatorname{sign}(\operatorname{Im} \lambda) = \frac{\pi}{2} \frac{\left(\lambda - \lambda^{-1}\right)}{\sqrt{4 - \left(\lambda + \lambda^{-1}\right)^2}}$$

$$= \frac{\pi}{4}\sum_{k=0}^{\infty}\frac{(2k)!}{2^{2k}(k!)^2}\left(\lambda-\frac{1}{\lambda}\right)\left[\frac{1}{2}\left(\lambda+\frac{1}{\lambda}\right)\right]^{2k}. \tag{72}$$

This implies that

$$\xi = -\frac{\pi}{4}\sum_{k=0}^{\infty}\frac{(2k)!}{2^{4k}(k!)^2}\mathcal{G}_k. \tag{73}$$

We see that the generator of $U(1)_L$ is indeed an infinite sum of local conserved charges, with only finitely many terms participating at each order of the null-surface perturbation theory.

Eq. (73) can also be obtained from Eqs. (44) and (70) using the formula

$$\lim_{\epsilon\to 0+}\sum_{p=0}^{\infty}\frac{e^{-2p\epsilon}}{2p+1}U_{2p}(x) = \frac{\pi}{4}(1-x^2)^{-1/2}. \tag{74}$$

5. Summary and discussion of $O(N)$ model

In this section we will summarize our construction of the action variable for the sine–Gordon model and outline the analogous construction for the $O(N)$ sigma-model.

5.1. The construction of the action variable for the sine–Gordon model

The sine–Gordon model has infinitely many local conserved charges, which are in involution with each other. Therefore each solution $\phi(\tau,\sigma)$ defines an infinite-dimensional "invariant torus" which is defined as the orbit of $\phi(\tau,\sigma)$ under the Hamiltonian flows generated by the local conserved charges. This "invariant torus" consists of the solutions $\phi(\tau,\sigma,t_3,\tilde{t}_3,t_5,\tilde{t}_5,\ldots)$ where t_{2n+1} and $\tilde{t}_{2n+1}$ are the "higher times" defined so that $\frac{\partial}{\partial t_{2n+1}}$ and $\frac{\partial}{\partial \tilde{t}_{2n+1}}$ are the Hamiltonian vector fields generated by the higher Hamiltonians. We define $t_1 = \frac{1}{4}(\tau+\sigma)$ and $\tilde{t}_1 = \frac{1}{4}(\tau-\sigma)$.

The Bäcklund transformations B_μ and $\tilde{B}_{\tilde{\mu}}$ depend on the parameters μ and $\tilde{\mu}$. They can be understood as the shifts of the higher times; B_μ is the shift $t_{2n+1}\to t_{2n+1}-\frac{\mu^{-2n-1}}{2n+1}$ and $\tilde{B}_{\tilde{\mu}}$ is the shift $\tilde{t}_{2n+1}\to\tilde{t}_{2n+1}+\frac{\tilde{\mu}^{2n+1}}{2n+1}$. We have:

$$\begin{aligned}&\frac{\partial}{\partial t_1}(B_\mu\phi+\phi) = -2\mu\sin(B_\mu\phi-\phi)\\&\frac{\partial}{\partial \tilde{t}_1}(B_\mu\phi-\phi) = \frac{2}{\mu}\sin(B_\mu\phi+\phi)\end{aligned} \tag{75}$$

$$\begin{aligned}&\frac{\partial}{\partial \tilde{t}_1}(\tilde{B}_{\tilde{\mu}}\phi+\phi) = \frac{2}{\tilde{\mu}}\sin(\tilde{B}_{\tilde{\mu}}\phi-\phi)\\&\frac{\partial}{\partial t_1}(\tilde{B}_{\tilde{\mu}}\phi-\phi) = -2\tilde{\mu}\sin(\tilde{B}_{\tilde{\mu}}\phi+\phi).\end{aligned} \tag{76}$$

We define B_μ as a power series in $1/\mu$ and $\tilde{B}_{\tilde{\mu}}$ as a power series in $\tilde{\mu}$. Then we define the "logarithm" of the Bäcklund transformation. For each μ (large) and $\tilde{\mu}$ (small) the Hamiltonian vector field $\xi_{\mu,\tilde{\mu}}$ is generated by a linear combination of the local Hamiltonians, such that $e^{\xi_{\mu,\tilde{\mu}}} = B_\mu\tilde{B}_{\tilde{\mu}}^{-1}$. The explicit formula is $\xi_{\mu,\tilde{\mu}} = -\sum\left[\frac{\mu^{-2p-1}}{2p+1}\frac{\partial}{\partial t_{2p+1}}+\frac{\tilde{\mu}^{2p+1}}{2p+1}\frac{\partial}{\partial \tilde{t}_{2p+1}}\right]$. We defined

B_μ perturbatively around $\mu = \infty$ and $\tilde{B}_{\tilde{\mu}}$ perturbatively around $\tilde{\mu} = 0$. But we have seen that (at least on the rational solutions, which form a dense set) there is a well-defined limit $\xi = \lim_{\mu\to 1,\tilde{\mu}\to 1} \xi_{\mu,\tilde{\mu}}$. Eqs. (75) and (76) suggest that $B_1\tilde{B}_1^{-1}$ is the transformation bringing ϕ to $-\phi$, and we have seen that this is indeed the case for rational solutions and in the null-surface perturbation theory. Therefore $\mathrm{e}^{2\xi}$ is the identical transformation, which means that ξ is generated by an action variable.

5.2. *The $O(N)$ sigma-model*

The same arguments can be applied to the $O(N)$ sigma-model. The Bäcklund transformations are defined by the same Eqs. (35) and (37) as for $N = 3$, but instead of a three-dimensional vector $\vec{n}(\tau,\sigma)$ we have an N-dimensional vector X:

$$\begin{aligned}\partial_{\tilde{t}_1}(B_\mu X - X) &= -\frac{1}{2}(1+\mu^{-2})(B_\mu X, \partial_{\tilde{t}_1} X)(B_\mu X + X)\\ \partial_{t_1}(B_\mu X + X) &= \frac{1}{2}(1+\mu^{2})(B_\mu X, \partial_{t_1} X)(B_\mu X - X)\end{aligned} \tag{77}$$

$$\begin{aligned}\partial_{t_1}(\tilde{B}_{\tilde{\mu}} X - X) &= -\frac{1}{2}(1+\tilde{\mu}^{2})(\tilde{B}_{\tilde{\mu}} X, \partial_{t_1} X)(\tilde{B}_{\tilde{\mu}} X + X)\\ \partial_{\tilde{t}_1}(\tilde{B}_{\tilde{\mu}} X + X) &= \frac{1}{2}(1+\tilde{\mu}^{-2})(\tilde{B}_{\tilde{\mu}} X, \partial_{\tilde{t}_1} X)(\tilde{B}_{\tilde{\mu}} X - X).\end{aligned} \tag{78}$$

We conjecture that the Bäcklund transformations correspond to the shift of times if we define them perturbatively as series in $1/\mu$ and $\tilde{\mu}$. (We do not know a proof of this fact for the $O(N)$ model.) But we can also define the Bäcklund transformations perturbatively using the null-surface perturbation theory. In the null-surface perturbation theory the small parameter is $1/|p| = 1/|\partial_\tau X|$ and $\partial_\sigma X$ remains finite. In the limit $|p| \to \infty$ we observe that (77) and (78) are solved by:

$$B_\mu X = \frac{1-\mu^{-2}}{1+\mu^{-2}} X - \frac{2\mu^{-1}}{1+\mu^{-2}} \frac{\partial_\tau X}{|\partial_\tau X|} \tag{79}$$

$$\tilde{B}_{\tilde{\mu}} X = \frac{1-\tilde{\mu}^{2}}{1+\tilde{\mu}^{2}} X + \frac{2\tilde{\mu}}{1+\tilde{\mu}^{2}} \frac{\partial_\tau X}{|\partial_\tau X|}. \tag{80}$$

The corrections to (79) and (80) by the higher powers of $1/|\partial_\tau X|$ involve higher derivatives in τ and σ and depend on μ as rational functions. When μ is large we can expand these corrections in powers of μ^{-1}. Therefore the definition of the Bäcklund transformation B_μ as a power series in $1/|\partial_\tau X|$ agrees with the usual definition as a power series in μ^{-1}, but does not require μ to be large.

As we did for the sine–Gordon model, we can define ξ_μ as the Hamiltonian vector field $\log B_\mu$ and $\tilde{\xi}_{\tilde{\mu}} = \log B_{\tilde{\mu}}$. Now we want to put $\mu = \tilde{\mu}$. There is a potential problem here, because B_μ was defined as a series in $1/\mu$ and $\tilde{B}_{\tilde{\mu}}$ as a series in $\tilde{\mu}$. But as we discussed, we can also define B_μ and $\tilde{B}_{\tilde{\mu}}$ in the null-surface perturbation theory using $1/|\partial_\tau X|$ as a small parameter. Then there is no problem doing the analytical continuation to $\mu = \tilde{\mu}$, because at every order of the $1/|\partial_\tau X|$ perturbation theory B_μ is a rational function of μ. (For example, the zeroth order is given by (79).) Eqs. (77)–(80) imply that for $\mu = \tilde{\mu}$ we have $B_\mu \tilde{B}_\mu^{-1} X = -X$. Therefore

$$\mathrm{e}^{\xi_\mu - \tilde{\xi}_\mu} = [X \mapsto -X].$$

This implies that the Hamiltonian vector field $\xi_\mu - \tilde{\xi}_\mu$ is generated by an action variable. This vector field is independent of the choice of μ (in the perturbation theory in $1/|\partial_\tau X|$) because of the uniqueness of the action variable. For the explicit calculation it would be convenient to choose $\mu = 1$, because with this choice it is manifest that the action variable is a combination of the local charges which is left–right symmetric. The vector field $\frac{\partial}{\partial\mu}(\xi_\mu - \tilde{\xi}_\mu)$ is zero in the perturbation theory because of the relations between the left and right charges discussed in Section 3 of [17].

Note in the revised version: see [38] for a discussion of the $O(N)$ model.

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